Asymptotic dependence of bivariate maxima

Helena Ferreira

Universidade da Beira Interior, Centro de Matemática e Aplicações (CMA-UBI), Avenida Marquês d'Avila e Bolama, 6200-001 Covilhã, Portugal

helena.ferreira@ubi.pt

Marta Ferreira

Center of Mathematics, University of Minho, Portugal

Center for Computational and Stochastic Mathematics, University of Lisbon, Portugal

Center of Statistics and Applications, University of Lisbon, Portugal

msferreira@math.uminho.pt

Abstract: The Ledford and Tawn model for the bivariate tail incorporates a coefficient, η, as a measure of pre-asymptotic dependence between the marginals. However, in the limiting bivariate extreme value model, G, of suitably normalized component-wise maxima, it is just a shape parameter without reflecting any description of the dependency in G. Under some local dependence conditions, we consider an index that describes the pre-asymptotic dependence in this context. We analyze some particular cases considered in the literature and illustrate with examples. A small discussion on inference is presented at the end.

Keywords: extreme value theory, stationary sequences, asymptotic dependence, dependence conditions

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1 Introduction

Consider a stationary sequence $\{(X_n, Y_n)\}_{n\geq 1}$ with distribution function (df) F belonging to the maximum domain of attraction of a bivariate extreme values (BEV) df G. The marginals of G, G_X and G_Y, are also extreme value df's and attract the maximum of $\{X_n\}$ and $\{Y_n\}$, respectively. The central result of the univariate extreme values theory, called Extremal Types Theorem, establishes the three possible limiting extreme value df's of the suitably normalized maximum of an independent and identically distributed (i.i.d.) sequence. This result was extended to stationary sequences under a distributional mixing condition D which states that the variables tend to independence as they get apart in time (Leadbetter *et al.* [12] 1983 and references therein).

The degree of dependence between G_X and G_Y can be evaluated through the extremal coefficient, $\varepsilon \in [1,2]$ (Tiago de Oliveira, [25] 1962-1963; Smith, [23] 1990), such that

$$P(G_X(X) \leq u, G_Y(Y) \leq u) = u^{\varepsilon},\ u \in [0,1],$$

assuming that the random pair (X,Y) has df G. Sufficient conditions to have $\varepsilon = 2$, that is, independence between $M_n^{(1)} = \max_{i=1}^n X_i$ and $M_n^{(2)} = \max_{i=1}^n Y_i$, suitably normalized, were presented in literature, both in the case of no clustering of high values within $\{X_n\}$ and $\{Y_n\}$ (Davis, [3] 1982), as well as, in the case that such clustering is allowed (Pereira, *et al.* [19] 2017). This latter scenario means that extreme events tend to occur in groups. The extremal index (Leadbetter *et al.* [12] 1983), usually denoted θ, measures the tendency for data to form clusters. Whenever $\theta = 1$, the extreme values tend to occur isolated and is a form of asymptotic independence. This may mean that either the data are independent, or there is eventually a residual dependence that vanishes as n tends to infinity and thus, in the limit, leading to the occurrence of isolated extremes. As far as we know, there is no discussion about this pre-asymptotic dependence in neither of these cases, i.e., the dependence between $M_n^{(1)}$ and $M_n^{(2)}$ with large n in concomitance with the independence between G_X and G_Y.

The topic of pre-asymptotic dependence, also denoted asymptotic independence, is assigned in the model of Ledford and Tawn (Ledford and Tawn, [14, 15] 1996/1997), in which we base our formulation of the joint right tail of (X_i, Y_j). More precisely, for $\tau_1, \tau_2 > 0$, and denoting $f_n \sim g_n$ whenever $f_n/g_n \to a \neq 0$, as $n \to \infty$, we consider

$$nP\left(X_i > \frac{n}{\tau_1}, Y_j >, \frac{n}{\tau_2}\right) \sim n^{-(1/\eta_{ij}-1)} \mathcal{L}_{\eta_{ij}}\left(\frac{n}{\tau_1}, \frac{n}{\tau_2}\right), \tag{1}$$

$i,j = 1,\ldots,n$, where $\eta \equiv \eta_{i,j} \in (0,1]$ and $\mathcal{L} \equiv \mathcal{L}_{\eta_{ij}}$ is a slowly varying function, i.e., there exists g such that, $\forall x, y > 0$ and $c > 0$,

$$g(x,y) = \lim_{t\to\infty} \frac{\mathcal{L}(tx,ty)}{\mathcal{L}(t,t)} \text{ and } g(cx,cy) = g(x,y). \tag{2}$$

We have asymptotic independence if $\eta < 1$ or if $\eta = 1$ and $\mathcal{L}\left(\frac{n}{\tau_1}, \frac{n}{\tau_2}\right) \to 0$, as $n \to \infty$, and tail dependence if $\eta = 1$ and $\mathcal{L}\left(\frac{n}{\tau_1}, \frac{n}{\tau_2}\right) \to a > 0$. The variables X_i and Y_j are (almost) independent if $\eta = 1/2$ and positively and negatively associated whenever $\eta > 1/2$ and $\eta < 1/2$, respectively. Ledford and Tawn ([14] 1996) showed that problems arise in modeling and inference if a pre-asymptotic dependence takes place and is ignored. See also Bortot and Tawn ([1] 1998) and Poon *et al.* ([20] 2003).

Suppose, without loss of generalization, that F has standard Fréchet marginals F_X and F_Y, and thus also G_X and G_Y. The Ledford and Tawn (Ledford and Tawn, [14, 15] 1996/1997) model assumption for the bivariate tail of G, which is given by

$$\overline{G}(u,u) = 1 - 2u + u^{\varepsilon} = (1-u)(2-\varepsilon) + (1-u)^2\varepsilon(\varepsilon-1)/2 + o((1-u)^2), \text{ as } u \uparrow 1,$$

would take us to $\eta = 1/2$ when $\varepsilon = 2$. Therefore, in this case, η cannot be interpreted as a pre-asymptotic dependence coefficient as in other df's which are not BEV. On the other hand, the Ledford and Tawn assumption to model the tail of F, although it allows interpreting η as a coefficient of pre-asymptotic dependence between the marginals F_X and F_Y, it appears in G, after suitable normalization of $M_n^{(1)}$ and $M_n^{(2)}$, as a shape parameter (Ramos and Ledford, [21] 2011) without expression in the description of the dependence of G.

Here, we discuss the conditions about the modeling in (1) that will lead to dependence between the marginals of G or to independence, describing in this case the type of pre-asymptotic dependence. On the local behavior of each marginal sequence $\{X_n\}$ and $\{Y_n\}$, we will assume that they satisfy Chernick *et al.* ([2] 1991) conditions, $D^{(s)}(u_n)$ and $D^{(t)}(v_n)$, for some $s \geq 1$ and $t \geq 1$, allowing clusters of extremes separated at least s and t, respectively, and together satisfy a local condition $D^{(k)}(u_n, v_n)$ regulating the joint location of clusters. A new index encompassing all types of asymptotic dependence between $M_n^{(1)}$ and $M_n^{(2)}$ will be presented in Section 2. In Section 3 we analyze the possible forms of pre-asymptotic dependence between $M_n^{(1)}$ and $M_n^{(2)}$ on some particular cases considered in the literature, along with illustrative examples. A discussion on Section 4 gives some insight about possible inference in this framework.

2 Index of asymptotic dependence between $M_n^{(1)}$ and $M_n^{(2)}$

Consider $\{(X_n, Y_n)\}$ a stationary sequence with standard Fréchet marginals and, for $\{(u_n, v_n)\}$ such that $n(1 - F_X(u_n)) \to \tau_1 > 0$ and $n(1 - F_Y(v_n)) \to \tau_2 > 0$, as $n \to \infty$, it is valid the condition $D(u_n, v_n)$ of Hsing ([11] 1989), meaning that $\alpha_{n,l_n} \to 0$ for some $l_n = o(n)$, as $n \to \infty$, where

$$\begin{aligned}\alpha_{n,l} = \max\{&|P(\max\nolimits_{i\in A} X_i \leq u_n, \max\nolimits_{i\in B} Y_i \leq v_n) - P(\max\nolimits_{i\in A} X_i \leq u_n)P(\max\nolimits_{i\in B} Y_i \leq v_n)| \\ &: A \subset \{1,2,\ldots,j\},\ B \subset \{j+l, j+l+1, \ldots, n\},\ 1 \leq j \leq n-l\},\ n \geq 1,\ 1 \leq l \leq n-1.\end{aligned} \quad (3)$$

Condition $D(u_n, v_n)$ extends the univariate distributional mixing condition D in Leadbetter *et al.* ([12] 1983) to the bivariate case and thus also allows to extend the Extremal Types Theorem to a stationary sequence of random vectors (Hsing, [11] 1989).

Furthermore, regarding the local behavior of each marginal sequence, we assume that $\{X_n\}$ satisfies the Chernick *et al.* ([2] 1991) dependence condition $D^{(s)}(u_n)$, for some $s \geq 1$, i.e.,

$$nP\left(X_1 > u_n, M_{2,s}^{(1)} \leq u_n < M_{s+1,r_n}^{(1)}\right) \to 0, \text{ as } n \to \infty, \tag{4}$$

where $M_{i,j}^{(1)} = \max_{l \in \{i,\dots,j\}} X_l$ with $\max_{l \in \{i,\dots,j\}} X_l = -\infty$ if $i > j$ and $r_n = [n/k_n]$ for some $\{k_n\}$ such that

$$k_n l_n/n \to 0,\ k_n/n \to 0,\ k_n \alpha_{n,l_n} \to 0. \tag{5}$$

Likewise we use notation $M_{i,j}^{(2)} = \max_{l \in \{i,\dots,j\}} Y_l$, with $\max_{l \in \{i,\dots,j\}} Y_l = -\infty$ if $i > j$. $\{Y_n\}$ satisfies $D^{(t)}(v_n)$, for some $t \geq 1$, with the same sequence $\{k_n\}$, without loss of generality. Both conditions allow clusters of exceedances of u_n and v_n, for $\{X_n\}$ and $\{Y_n\}$, respectively, separated at least $s \geq 1$ and $t \geq 1$. Concerning the joint location of the clusters of $\{X_n\}$ and $\{Y_n\}$, we admit that they are distant from each other at most $k \geq 0$, i.e.,

$$k_n \sum_{\substack{i=1 \\ |i-j|>k}}^{r_n} \sum_{j=1}^{r_n} P\left(X_i > u_n, M_{i+1,i+s-1}^{(1)} \leq u_n, Y_j > v_n, M_{j+1,j+t-1}^{(2)} \leq v_n\right) \to 0, \text{ as } n \to \infty. \tag{6}$$

This condition will be denoted $D^{(k)}(u_n, v_n)$ and simplifies the description of the dependence between G_X and G_Y through the asymptotic behavior of the joint tail of X_i and Y_j for a finite number of pairs (i,j). Observe that the simpler statement

$$k_n \sum_{\substack{i=1 \\ |i-j|>k}}^{r_n} \sum_{j=1}^{r_n} P\left(X_i > u_n, Y_j > v_n\right) \to 0, \text{ as } n \to \infty, \tag{7}$$

implies $D^{(k)}(u_n, v_n)$ in (6) and thus can be used for checking the validity of this latter.

Lemma 2.1. *If $\{(X_n, Y_n)\}$ satisfies condition $D(u_n, v_n)$ in (3) for coefficients $\{\alpha_n, l_n\}$, $\{X_n\}$ satisfies $D^{(s)}(u_n)$, $\{Y_n\}$ satisfies $D^{(t)}(v_n)$ and $\{(X_n, Y_n)\}$ satisfies $D^{(k)}(u_n, v_n)$ for some $\{k_n\}$ satisfying (5), then*

$$\begin{aligned}\lim_{n\to\infty} P\left(M_n^{(1)} \leq u_n, M_n^{(2)} \leq v_n\right) = \quad & \exp\left\{-\lim_{n\to\infty} nP\left(X_1 > u_n \geq M_{2,s}^{(1)}\right) - nP\left(Y_1 > v_n \geq M_{2,t}^{(2)}\right)\right. \\ & \left. + \lim_{n\to\infty} \sum_{j=0}^{2k} nP\left(X_{k+1} > u_n \geq M_{k+2,k+s}^{(1)}, Y_{j+1} > v_n \geq M_{j+2,j+t}^{(2)}\right)\right\}.\end{aligned}$$

Proof. From condition $D(u_n, v_n)$ and the stationarity assumption, we have (Hsing [11] 1989; Lemma

4.1),

$$\begin{aligned}
\lim_{n\to\infty} P\left(M_n^{(1)} \le u_n, M_n^{(2)} \le v_n\right) &= \lim_{n\to\infty} P^{k_n}\left(M_{r_n}^{(1)} \le u_n, M_{r_n}^{(2)} \le v_n\right) \\
&= \lim_{n\to\infty} \left(1 - \frac{k_n P\left(\{M_{r_n}^{(1)} > u_n\} \cup \{M_{r_n}^{(2)} > v_n\}\right)}{k_n}\right)^{k_n} \\
&= \exp\left\{-\lim_{n\to\infty} k_n P\left(\{M_{r_n}^{(1)} > u_n\} \cup \{M_{r_n}^{(2)} > v_n\}\right)\right\} .
\end{aligned}$$

Under conditions $D^{(s)}(u_n)$ for $\{X_n\}$ and $D^{(t)}(v_n)$ for $\{Y_n\}$, we have that (Chernick *et al.* [2] 1991; Proposition 1.1 and references therein)

$$\lim_{n\to\infty} k_n P\left(M_{r_n}^{(1)} > u_n\right) = \lim_{n\to\infty} nP\left(X_1 > u_n, X_2 \le u_n, \dots, X_s \le u_n\right)$$

and

$$\lim_{n\to\infty} k_n P\left(M_{r_n}^{(2)} > v_n\right) = \lim_{n\to\infty} nP\left(Y_1 > v_n, Y_2 \le v_n, \dots, Y_t \le v_n\right) .$$

In what follows, we apply a commonly used extreme values technique that consists in omitting terms which summation converges to zero, as $n \to \infty$, under the validity of dependence conditions (see, e.g. Leadbetter and Nandagopalan [13] 1989). More precisely, under $D^{(k)}(u_n, v_n)$ and the stationarity,

$$
\begin{aligned}
& \lim_{n\to\infty} k_n P\left(M_{r_n}^{(1)} > u_n, M_{r_n}^{(2)} > v_n\right) \\
= \quad & \lim_{n\to\infty} k_n \sum_{i=1}^{r_n}\sum_{j=1}^{r_n} P\left(X_i > u_n, M_{i+1,r_n}^{(1)} \le u_n, Y_j > v_n, M_{j+1,r_n}^{(2)} \le v_n\right) \\
= \quad & \lim_{n\to\infty} k_n \sum_{\substack{i=1\\|i-j|\le k}}^{r_n}\sum_{j=1}^{r_n} P\left(X_i > u_n, M_{i+1,r_n}^{(1)} \le u_n, Y_j > v_n, M_{j+1,r_n}^{(2)} \le v_n\right) \\
= \quad & \lim_{n\to\infty} k_n \sum_{\substack{i=1\\|i-j|\le k}}^{r_n}\sum_{j=1}^{r_n} P\left(X_1 > u_n, M_{2,r_n-i+1}^{(1)} \le u_n, Y_{j-i+1} > v_n, M_{j-i+2,r_n-i+1}^{(2)} \le v_n\right) \\
= \quad & \lim_{n\to\infty} k_n \sum_{i=1}^{r_n}\sum_{j=-k}^{k} P\left(X_1 > u_n, M_{2,r_n-i+1}^{(1)} \le u_n, Y_{j+1} > v_n, M_{j+2,r_n-i+1}^{(2)} \le v_n\right) \\
= \quad & \lim_{n\to\infty} k_n \sum_{i=1}^{r_n}\sum_{j=-k}^{k} P\left(X_1 > u_n, M_{2,r_n}^{(1)} \le u_n, Y_{j+1} > v_n, M_{j+2,r_n}^{(2)} \le v_n\right) \\
= \quad & \lim_{n\to\infty} \sum_{j=-k}^{k} nP\left(X_1 > u_n, M_{2,r_n}^{(1)} \le u_n, Y_{j+1} > v_n, M_{j+2,r_n}^{(2)} \le v_n\right) .
\end{aligned}
$$

By applying again conditions $D^{(s)}(u_n)$ for $\{X_n\}$ and $D^{(t)}(v_n)$ for $\{Y_n\}$, we conclude that the previous limit becomes

$$
\lim_{n\to\infty} \sum_{j=-k}^{k} nP\left(X_1 > u_n, M_{2,s}^{(1)} \le u_n, Y_{j+1} > v_n, M_{j+2,t}^{(2)} \le v_n\right) .
$$

□

For each $(\tau_1, \tau_2) \in \mathbb{R}_+^2$, the value

$$
\xi(\tau_1,\tau_2) = \lim_{n\to\infty} \sum_{j=0}^{2k} nP\left(X_{k+1} > u_n \ge M_{k+2,k+s}^{(1)}, Y_{j+1} > v_n \ge M_{j+2,j+t}^{(2)}\right) \ge 0, \tag{8}
$$

provided that the limit exists for $\{(u_n, v_n)\}$ such that $n(1 - F_X(u_n)) \to \tau_1 > 0$ and $n(1 - F_Y(v_n)) \to \tau_2 > 0$, as $n \to \infty$, appears as a quantifying parameter of the asymptotic dependence between $M_n^{(1)}$ and $M_n^{(2)}$. Once the local dependence conditions are validated, this index depends on the joint behavior of a finite number of the variables of the process. This index contemplates the possibility of joint occurrence of clusters of high values, for each sequence of margins separated by a maximum of $k \ge 0$. By assuming $D^{(s)}(u_n)$, $D^{(t)}(v_n)$ and $D^{(k)}(u_n, v_n)$, we do not establish any relation between s, t and k, that is, between the minimum distances separating clusters of the same sequence of margins (s and t)

and the maximum distance between clusters of distinct margins (k). In the following we state two more properties concerning function $\xi(\tau_1,\tau_2)$.

Proposition 2.2. *Under conditions of Lemma 2.1, if* $P\left(M_n^{(1)}\leq n/\tau_1, M_n^{(2)}\leq n/\tau_2\right)\to H(\tau_1^{-1},\tau_2^{-1})$, *as* $n\to\infty$ *and* $(\tau_1,\tau_2)\in\mathbb{R}_+^2$, *for some BEV df* H, *then function* $\xi(\tau_1,\tau_2)$ *is homogeneous of order 1 provided it is non-constant.*

Proof. By Corollary 1.3 in Chernick *et al.* ([2], 1991), we have that $P(M_{2,s}^{(1)}\leq u_n|X_1>u_n)\to\theta_X$, as well as $P(M_{2,t}^{(2)}\leq v_n|Y_1>v_n)\to\theta_Y$, where θ_X and θ_Y are the respective marginal extremal indexes. Now, just observe that

$$
\begin{aligned}
P\left(M_n^{(1)}\leq \tfrac{n}{t\tau_1}, M_n^{(2)}\leq \tfrac{n}{t\tau_2}\right)\to\quad & e^{-\theta_X t\tau_1}e^{-\theta_Y t\tau_2}e^{\xi(t\tau_1,t\tau_2)}=H\left((t\tau_1)^{-1},(t\tau_2)^{-1}\right)=H^t(\tau_1^{-1},\tau_2^{-1})\\
&=\left(e^{-\theta_X\tau_1}e^{-\theta_Y\tau_2}e^{\xi(\tau_1,\tau_2)}\right)^t,
\end{aligned}
$$

where the second equality is due to a max-stability property of a BEV distribution (Galambos [10] 1987; Theorem 5.2.1). Thus $\xi(t\tau_1,t\tau_2)=t\xi(\tau_1,\tau_2)$. $\square$

Proposition 2.3. *Under conditions of Lemma 2.1, if* $\{(X_n,Y_n)\}$ *has bivariate extremal index* $\theta(\tau_1,\tau_2)$, *then*

$$
\theta(\tau_1,\tau_2)=\frac{\theta_X\tau_1+\theta_Y\tau_2-\xi(\tau_1,\tau_2)}{\tau_1+\tau_2-\lambda(\tau_1,\tau_2)}\,, \tag{9}
$$

where $\lambda(\tau_1,\tau_2)=\lim_{n\to\infty}nP(X_1>n/\tau_1,Y_1>n/\tau_2)$.

Proof. Since

$$
\lim_{n\to\infty}nP\left(\{X_1>n/\tau_1\}\cup\{Y_1>n/\tau_2\}\right)=\tau_1+\tau_2-\lim_{n\to\infty}nP\left(X_1>\frac{n}{\tau_1},Y_1>\frac{n}{\tau_2}\right)=\tau_1+\tau_2-\lambda(\tau_1,\tau_2),
$$

then

$$
P\left(M_n^{(1)}\leq \tfrac{n}{\tau_1}, M_n^{(2)}\leq \tfrac{n}{\tau_2}\right)\to\left(e^{-\theta_X\tau_1}e^{-\theta_Y\tau_2}e^{\xi(\tau_1,\tau_2)}\right)=\exp\{-\theta(\tau_1,\tau_2)(\tau_1+\tau_2-\lambda(\tau_1,\tau_2))\}
$$

with $\theta(\tau_1,\tau_2)$ satisfying (9). $\square$

Observe that $\lambda(\tau_1,\tau_2)$ above corresponds to the bivariate upper tail copula function considered in Schmidt and Stadtmüller ([22] 2006). See also Li ([17] 2009) and references therein. The bivariate extremal index was introduced in Nandagopalan [18] 1994. More recent developments can be seen in Pereira, *et al.* ([19] 2017).

If the marginals of the limiting BEV H are independent, we have $\xi(\tau_1, \tau_2) = 0$. However, a residual tail dependence measured through the rate of convergence of $\xi(\tau_1, \tau_2)$ towards zero may occur. This type of dependence is usually ruled in the literature through the Ledford and Tawn coefficient η, defined in (1). This is addressed in the next section.

3 Pre-asymptotic dependence between $M_n^{(1)}$ and $M_n^{(2)}$

We are going to analyze the asymptotic dependence function $\xi(\tau_1, \tau_2)$ in (8), by considering two particular cases for s and t often addressed in the literature.

Proposition 3.1. *Under conditions of Lemma 2.1, if $s = t = 1$ and, as $n \to \infty$,*

$$nP(X_i > u_n, Y_j > v_n) \sim n^{-(1/\eta_{ij}-1)} \mathcal{L}_{\eta_{ij}} \left(\frac{n}{\tau_1}, \frac{n}{\tau_2} \right) \tag{10}$$

holds for all $j = 1, \ldots, 2k+1$ and $i = k+1$, with $\eta_{ij} \equiv \eta_{ij}(\tau_1, \tau_2) \in (0, 1]$ and $\mathcal{L}_{\eta_{ij}}$ slowly varying functions, then

$$\xi(\tau_1, \tau_2) \sim n^{-(1/\eta-1)} \mathcal{L}^* \left(\frac{n}{\tau_1}, \frac{n}{\tau_2} \right), \tag{11}$$

where $\eta = \max\{\eta_{ij} : j = 1, \ldots, 2k+1,\, i = k+1\}$ and

$$\mathcal{L}^* \left(\frac{n}{\tau_1}, \frac{n}{\tau_2} \right) = \sum_{j=0}^{2k} n^{-(1/\eta_{ij}-1/\eta)} \mathcal{L}_{\eta_{ij}} \left(\frac{n}{\tau_1}, \frac{n}{\tau_2} \right)$$

is a slowly varying function.

Proof. Under conditions $D^{(1)}(u_n)$ and $D^{(1)}(v_n)$, we have $\theta_X = \theta_Y = 1$ (Chernick *et al.*, [2] 1991; Corollary 1.3). Now observe that,

$$\lim_{n\to\infty} P\left(M_n^{(1)} \le u_n, M_n^{(2)} \le v_n\right) = e^{-\nu_1} e^{-\nu_2} e^{\xi(\tau_1,\tau_2)}, \tag{12}$$

with $\nu_1 = \tau_1$, $\nu_2 = \tau_2$ and

$$\xi(\tau_1, \tau_2) = \lim_{n\to\infty} \sum_{j=0}^{2k} nP\left(X_{k+1} > u_n, Y_{j+1} > v_n\right), \tag{13}$$

for all $k \ge 0$. □

In the context of Proposition 3.1 we have *ξ-asymptotic tail independence* if $\eta < 1$ or if $\eta = 1$ and

$\mathcal{L}^*\left(\frac{n}{\tau_1},\frac{n}{\tau_2}\right) \to 0$, as $n \to \infty$ (which holds if $\mathcal{L}_{\eta_{ij}}\left(\frac{n}{\tau_1},\frac{n}{\tau_2}\right) \to 0$, for all $j = 1,\ldots,2k+1$, $i = k+1$, such that $\eta_{ij} = 1$). This case lead us to $\xi(\tau_1,\tau_2) = 0$.

We have ξ-*tail dependence* if $\eta = 1$ and $\mathcal{L}^*\left(\frac{n}{\tau_1},\frac{n}{\tau_2}\right) \to c > 0$, as $n \to \infty$ (which holds if $\mathcal{L}_{\eta_{ij}}\left(\frac{n}{\tau_1},\frac{n}{\tau_2}\right) \to c_j > 0$, for some $j = 1,\ldots,2k+1$, $i = k+1$, such that $\eta_{ij} = 1$). Now we obtain $\xi(\tau_1,\tau_2) > 0$.

Observe that, in order to have $\xi(\tau_1,\tau_2) = 0$, all random pairs (X_i,Y_j), $j = 1,\ldots,2k+1$, $i = k+1$, must be asymptotic tail independent. On the other hand, if one random pair is tail dependent then $\xi(\tau_1,\tau_2) > 0$. Notice also that this evaluation is based on exceedances of high thresholds. In the next case our analysis is based on down-crossings of extreme thresholds.

Proposition 3.2. *Under conditions of Lemma 2.1, if $s = t = 2$ and*

$$nP(X_i \geq u_n > X_{i+1}, Y_j \geq v_n > Y_{j+1}) \sim n^{-(1/\beta_{ij}-1)}\mathcal{L}_{\beta_{ij}}\left(\frac{n}{\tau_1},\frac{n}{\tau_2}\right) \tag{14}$$

holds, as $n \to \infty$, for all $j = 1,\ldots,2k+1$ and $i = k+1$, with $\beta_{ij} \equiv \beta_{ij}(\tau_1,\tau_2) \in (0,1]$ and $\mathcal{L}_{\beta_{ij}}$ slowly varying functions. Then

$$\xi(\tau_1,\tau_2) \sim n^{-(1/\beta-1)}\mathcal{L}^{**}\left(\frac{n}{\tau_1},\frac{n}{\tau_2}\right), \tag{15}$$

where $\beta = \max\{\beta_{ij} : j = 1,\ldots,2k+1,\, i = k+1\}$ and

$$\mathcal{L}^{**}\left(\frac{n}{\tau_1},\frac{n}{\tau_2}\right) = \sum_{j=0}^{2k} n^{-(1/\beta_{ij}-1/\beta)}\mathcal{L}_{\beta_{ij}}\left(\frac{n}{\tau_1},\frac{n}{\tau_2}\right) \tag{16}$$

is a slowly varying function. Moreover if we assume, as $n \to \infty$, that

$$nP\left(\bigcap_{i\in I}\{X_i > u_n\}, \bigcap_{j\in J}\{Y_j > v_n\}\right) \sim n^{-(1/\eta_{I,J}-1)}\mathcal{L}_{\eta_{I,J}}\left(\frac{n}{\tau_1},\frac{n}{\tau_2}\right), \tag{17}$$

for all $I \subseteq \{k+1,k+2\}$ and $J \subseteq \{1,\ldots,2k+2\}$, then $\beta = \max\{\eta_{ij} : j = 1,\ldots,2k+1,\, i = k+1\}$ and

$$\begin{aligned}\mathcal{L}_{\beta_{ij}}\left(\frac{n}{\nu_1},\frac{n}{\nu_2}\right) \sim\ & \mathcal{L}_{\eta_{ij}}\left(\frac{n}{\tau_1},\frac{n}{\tau_2}\right) - n^{-(1/\eta_{\{i\},\{j,j+1\}}-1)}\mathcal{L}_{\eta_{\{i\},\{j,j+1\}}}\left(\frac{n}{\tau_1},\frac{n}{\tau_2}\right)\\ & - n^{-(1/\eta_{\{i,i+1\},\{j\}}-1)}\mathcal{L}_{\eta_{\{i,i+1\},\{j\}}}\left(\frac{n}{\tau_1},\frac{n}{\tau_2}\right)\\ & + n^{-(1/\eta_{\{i,i+1\},\{j,j+1\}}-1)}\mathcal{L}_{\eta_{\{i,i+1\},\{j,j+1\}}}\left(\frac{n}{\tau_1},\frac{n}{\tau_2}\right),\end{aligned} \tag{18}$$

where $\eta_{ij} \equiv \eta_{\{i\},\{j\}}$.

Proof. Just notice that (12) holds with $\nu_1 = \tau_1\theta_1$, $\nu_2 = \tau_2\theta_2$, $\theta_1, \theta_2 \in (0,1]$ and

$$\xi(\tau_1,\tau_2) = \lim_{n\to\infty} \sum_{j=0}^{2k} nP\left(X_{k+1} \geq u_n > X_{k+2}, Y_{j+1} \geq v_n > Y_{j+2}\right), \tag{19}$$

for all $k \geq 0$.

The second part is straightforward from Proposition 2 of Ferreira and Ferreira ([7] 2012). $\square$

Observe that β_{ij} is similar to the up-crossings asymptotic tail independent coefficient introduced in Ferreira and Ferreira ([7] 2012). Analogously to the previous case, we can exploit tail (in)dependence under the point of view of down-crossings of high levels. Therefore, we have *ξ-asymptotic tail independence* if $\beta < 1$ or if $\beta = 1$ and $\mathcal{L}^{**}\left(\frac{n}{\tau_1}, \frac{n}{\tau_2}\right) \to 0$, as $n \to \infty$ (leading to $\xi(\tau_1,\tau_2) = 0$) and *ξ-tail dependence* if $\beta = 1$ and $\mathcal{L}^{**}\left(\frac{n}{\tau_1}, \frac{n}{\tau_2}\right) \to c > 0$, as $n \to \infty$ (obtaining $\xi(\tau_1,\tau_2) > 0$). Once again, in order to have $\xi(\tau_1,\tau_2) = 0$, all random pairs (X_i, Y_j), $j = 1,\ldots,2k+1$, $i = k+1$, must be down-crossings asymptotic tail independent, but if one random pair is down-crossings tail dependent then $\xi(\tau_1,\tau_2) > 0$.

Example 3.1. *Let $\{X_n^*\}$ and $\{Y_n^*\}$ be stationary sequences such that conditions $D^{(s)}(u_n)$ and $D^{(t)}(v_n)$ respectively hold, and $\{Z_n\}$ be an i.i.d. sequence independent of $\{(X_n^*, Y_n^*)\}$, all having common margin standard Fréchet. Consider*

$$X_n = X_n^* \vee Z_n^{1/\alpha} \quad \textit{and} \quad Y_n = Y_n^* \vee Z_n^{1/\rho}, \tag{20}$$

where $\alpha, \rho \in (0,1)$, corresponding to a pMAX model introduced in Ferreira and Ferreira ([5] 2014). We have that $\{X_n\}$ and $\{Y_n\}$ also satisfy conditions $D^{(s)}(u_n)$ and $D^{(t)}(v_n)$, respectively. Consider the particular case where $\{Y_n^* = X_n^* \mathbb{1}_{\{J_n=0\}} + X_{n+1}^* \mathbb{1}_{\{J_n=1\}}\}$, with $\{J_n\}$ an i.i.d. Bernoulli sequence and $s = t = 1$. We have $\theta_X = \theta_{X^*} = 1$, $\theta_Y = \theta_{Y^*} = 1$ (see Proposition 2.2 in Ferreira and Ferreira, [5] 2014) and $\xi(\tau_1,\tau_2)$ is given by (13). Assuming that, as $n \to \infty$,

$$nP\left(X_i^* > u_n, X_l^* > v_n\right) \sim n^{-(1/\eta_{i,l}^{(X^*)}-1)} \mathcal{L}_{\eta_{i,l}^{(X^*)}}\left(\frac{n}{\tau_1}, \frac{n}{\tau_2}\right),$$

for $i = k+1$ and $l = 1,\ldots,2k+2$, thus

$$\begin{aligned} nP\left(X_i^* > u_n, Y_j^* > v_n\right) = \; & nP(X_i^* > u_n, X_j^* > v_n)(1-p) + nP(X_i^* > u_n, X_{j+1}^* > v_n)p \\ \sim \; & n^{-(1/\eta_{i,j}^{(X^*)}-1)} \mathcal{L}_{\eta_{i,j}^{(X^*)}}\left(\tfrac{n}{\tau_1}, \tfrac{n}{\tau_2}\right) + n^{-(1/\eta_{i,j+1}^{(X^*)}-1)} \mathcal{L}_{\eta_{i,j+1}^{(X^*)}}\left(\tfrac{n}{\tau_1}, \tfrac{n}{\tau_2}\right) \\ \sim \; & n^{-(1/\eta_{i,j}^{(X^*,Y^*)}-1)} \mathcal{L}_{\eta_{i,j}^{(X^*,Y^*)}}\left(\tfrac{n}{\tau_1}, \tfrac{n}{\tau_2}\right), \end{aligned}$$

where, for $i = k+1$ and $j = 1,\ldots,2k+1$, $\eta_{i,j}^{(X^*,Y^*)} = \max\{\eta_{i,j}^{(X^*)}, \eta_{i,j+1}^{(X^*)}\} = 1$, since $\eta_{k+1,k+1}^{(X^*)} = 1$ and *thus* $\eta_{k+1,k+1}^{(X^*,Y^*)} = 1$.

Therefore, by applying Proposition 2.6 in Ferreira and Ferreira ([5] 2014)), we have that (11) holds with

$$\begin{aligned} \eta = & \max\left\{\tfrac{\alpha}{\alpha+\min\{1,\rho\}}, \alpha\eta_{i,j}^{(X^*,Y^*)} : i = k+1,\, j = 1,\ldots,2k+1\right\} \\ = & \max\left\{\tfrac{\alpha}{\alpha+\min\{1,\rho\}}, \alpha : i = k+1,\, j = 1,\ldots,2k+1\right\}. \end{aligned}$$

Example 3.2. *Consider again the pMAX model above in (20), where* $\alpha, \rho \in [1,\infty)$. *Consider the particular case where* $k = 1$, *and* $s = t = 2$, $\{X_n^*\}$ 1*-dependent (and thus satisfy* $D^{(2)}(u_n)$*) and* $\{Y_n^* = X_{n+3}^*\mathbb{1}_{\{J_n=0\}} + X_{n+4}^*\mathbb{1}_{\{J_n=1\}}\}$, *with* $\{J_n\}$ *an i.i.d. Bernoulli sequence.We have* $\nu_1 = \theta_X = \theta_{X^*}$, $\nu_2 = \theta_Y = \theta_{Y^*}$ *(see Proposition 2.2 in Ferreira and Ferreira, [5] 2014) and*

$$\begin{aligned} \xi(\tau_1,\tau_2) = & nP\left(X_2 > u_n \geq X_3, Y_1 > v_n \geq Y_2\right) + nP\left(X_2 > u_n \geq X_3, Y_2 > v_n \geq Y_3\right) \\ & + nP\left(X_2 > u_n \geq X_3, Y_3 > v_n \geq Y_4\right). \end{aligned}$$

Since $\{X_n^*\}$ 1*-dependent, as* $n \to \infty$, *we have*

$$nP\left(X_2^* > u_n, X_j^* > v_n\right) \sim \frac{\tau_1\tau_2}{n}$$

for $j \geq 4$, *and thus* $\eta_{2,j}^{(X^*,Y^*)} = 1/2$.

By Proposition 2.6 in Ferreira and Ferreira ([5] 2014)), we have that (15) holds with

$$\beta = \max\left\{\frac{1}{\alpha+1}, \frac{1}{\rho+1}, \frac{1}{2}\right\}.$$

The example below addresses factor models, used in the modeling of large losses within, e.g., insurance (Lescourret and Robert, [16] 2006) and finance (Ferreira and Canto e Castro, [8] 2010; Ferreira and Ferreira, [4] 2015). See also Li ([17], 2009) and references therein.

Example 3.3. *Consider the mixture model,* $(X_n, Y_n) = (RX_n^*, RY_n^*)$, *where sequences* $\{X_n^*\}$ *and* $\{Y_n^*\}$ *satisfy, respectively, conditions* $D^{(s)}(u_n)$ *and* $D^{(t)}(v_n)$ *and have extremal indexes* θ_{X^*} *and* θ_{Y^*}, *and where* R *is a positive r.v. independent of* $\{(X_n^*, Y_n^*)\}$ *and such that* $E(R) < \infty$. *If* $\{(X_n^*, Y_n^*)\}$ *satisfies* $D^{(k)}(u_n, v_n)$ *then* $\{(X_n, Y_n)\}$ *satisfies it as well. Let* $u_n^* = n/\tau_1^*$ *and* $v_n^* = n/\tau_2^*$ *be normalized levels for* $\{X_n^*\}$ *and* $\{Y_n^*\}$. *Thus, they are normalized levels for* $\{X_n\}$ *and* $\{Y_n\}$ *with* $\tau_1 = E(R)\tau_1^*$ *and*

$\tau_2 = E(R)\tau_2^*$, respectively. By applying (8), we have

$$\xi(\tau_1,\tau_2) = \lim_{n\to\infty}\int_0^\infty \sum_{j=0}^{2k} nP\left(X_{k+1}^* > \frac{n}{\tau_1^* r} \geq M_{k+2,k+s}^{(1)}, Y_{j+1}^* > \frac{n}{\tau_2^* r} \geq M_{j+2,j+t}^{(2)}\right) dF_R(r)$$

$$= \lim_{n\to\infty}\int_0^\infty \xi^*(\tau_1^* r, \tau_2^* r) dF_R(r) = \xi^*(\tau_1^*, \tau_2^*)E(R),$$

if $\xi^*(\tau_1^*, \tau_2^*)$ exists and is homogeneous of order 1. Assuming that, as $n \to \infty$,

$$nP\left(X_i^* > \frac{n}{\tau_1^*}, Y_j^* > \frac{n}{\tau_2^*}\right) \sim n^{-(1/\eta_{ij}^*-1)}\mathcal{L}_{\eta_{ij}^*}\left(\frac{n}{\tau_1^*}, \frac{n}{\tau_2^*}\right),$$

we have, by applying the dominated convergence theorem,

$$\begin{aligned} nP\left(RX_i^* > \tfrac{n}{\tau_1^*}, RY_j^* > \tfrac{n}{\tau_2^*}\right) &= \int_0^\infty nP\left(X_i^* > \frac{n}{\tau_1^* r}, Y_j^* > \frac{n}{\tau_2^* r}\right) dF_R(r) \\ &\sim \int_0^\infty r^{1/\eta_{ij}^*} n^{-(1/\eta_{ij}^*-1)}\mathcal{L}_{\eta_{ij}^*}\left(\frac{n}{\tau_1^* r}, \frac{n}{\tau_2^* r}\right) dF_R(r) \\ &\sim \int_0^\infty r^{1/\eta_{ij}^*} n^{-(1/\eta_{ij}^*-1)}\mathcal{L}_{\eta_{ij}^*}\left(\frac{n}{\tau_1^*}, \frac{n}{\tau_2^*}\right) dF_R(r) \\ &= n^{-(1/\eta_{ij}^*-1)}\mathcal{L}_{\eta_{ij}^*}\left(\tfrac{n}{\tau_1^*}, \tfrac{n}{\tau_2^*}\right) E(R^{1/\eta_{ij}^*}), \end{aligned}$$

provided $E(R^{1/\eta_{ij}^*})$ exists. Thus, we can state

$$nP\left(X_i > \frac{n}{\tau_1}, Y_j > \frac{n}{\tau_2}\right) \sim n^{-(1/\eta_{ij}-1)}\mathcal{L}_{\eta_{ij}}\left(\frac{n}{\tau_1}, \frac{n}{\tau_2}\right),$$

where $\eta_{ij} = \eta_{ij}^*$ and $\mathcal{L}_{\eta_{ij}}\left(\frac{n}{\tau_1}, \frac{n}{\tau_2}\right) = \mathcal{L}_{\eta_{ij}^*}\left(\frac{n}{\tau_1^*}, \frac{n}{\tau_2^*}\right) E(R^{1/\eta_{ij}^*})$.

4 Discussion

In this paper we introduce a new index, $\xi(\tau_1, \tau_2)$, in order to measure a (pre-)asymptotic dependence between the component-wise maxima of a bivariate stationary sequence. We consider the marginal local behavior of the sequence ruled through Chernick *et al.* ([2] 1991) dependence conditions, $D^{(s)}(u_n)$ and $D^{(t)}(v_n)$, for some $s, t > 0$, along with a bivariate local dependence condition $D^{(k)}(u_n, v_n)$, $k > 0$, defined here. An empirical approach to validate some $D^{(s)}(u_n)$ was presented in Ferreira and Ferreira ([6] 2016). See also Süveges ([24] 2007). An automated statistical method for joint selection of threshold

u_n and parameter s can be seen in Fukutome *et al.* ([9] 2014). We believe that both methodologies can be extended to $D^{(k)}(u_n, v_n)$, at least through condition (7). In Ledford and Tawn ([15] 1997) we can find parametric estimation based on maximum likelihood (and thus not suitable in our context which assumes dependence between random pairs), as well as, a non-parametric proposal. This approach will be a starting point to address this topic in a future work.

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